Chapter 5.5: Indefinite Integrals and the Substitution Method

Substitution Method

Helping to integrate chain rule

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x) \qquad \int f'(g(x))g'(x)\,dx = f(g(x)) + C$$

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du \qquad \int 2x\sqrt{x^2+1}\,dx = \int \sqrt{u}\,du$$

$$u = g(x) \qquad u = x^2 + 1$$

$$du = g'(x)\,dx \qquad du = 2x\,dx$$

Goal is to simplify the expression.
Look for a function in a function to identify the substitution to make.
Sometimes need to write $dx = h(u)\,du$.

$$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x^2+1)^{3/2} + C$$

Substitution examples

$$\int 3x^2 e^{x^3}\, dx$$

$$\int \sin(7\theta + 3)\, d\theta$$

$$\int (3x^2 + 1)(x^3 + x)^4\, dx$$

$$\int \frac{z}{\sqrt[3]{z^2 + 1}}\, dz$$

$$\int (x^6 + x)^4\, dx$$

$$\int \sin(x)^3 \cos(x)^3\, dx$$

$$\int \frac{1}{e^x + e^{-x}}\, dx$$

$\int (1 + \sqrt[3]{x})^a\, dx$ for $a \neq -1, -2, -3$.

$$\int 3x^2 e^{x^3}\ dx = \int e^u\ du = e^u + C = e^3 + C$$

$$u = x^3$$

$$du = 3x^2\ dx$$

$$\int \sin(7\theta+3)\ d\theta = \int \sin(u)\frac{1}{7}\ du = -\frac{1}{7}\cos(u) + C = -\frac{1}{7}\cos(7\theta+3) + C.$$

$$u = 7\theta + 3$$

$$du = 7\ d\theta$$

$$\int (3x^2+1)(x^3+x)^4\ dx = \int u^4\ du = \frac{1}{5}u^5 + C = \frac{1}{5}(x^3+x)^5 + C$$

$$u = x^3 + x$$

$$du = 3x^2 + 1\ dx$$

$$\int \frac{z}{\sqrt[3]{z^2+1}}\,dz = \int \frac{1}{2}\frac{1}{\sqrt[3]{u}}\,du = \frac{1}{2}\frac{3}{2}u^{2/3} + C = \frac{3}{4}(z^2+1)^{2/3} + C$$

$$u = z^2 + 1$$
$$du = 2z\,dz$$

$$\int (x^6 + x)^4\,dx = \int x^4(x^5+1)^4\,dx = \int \frac{1}{5}u^4\,du = \frac{1}{25}u^5 + C = \frac{1}{25}(x^5+1)^5 + C$$

$$u = x^5 + 1$$
$$du = 5x^4\,dx$$

$$
\begin{aligned}
\int \sin(x)^3 \cos(x)^3 \, dx &= \int sin(x) \cdot (1 - cos(x)^2) \cdot cos(x)^3 \, dx \\
&= \int \sin(x) \cos(x)^3 - \sin(x) \cos(x)^5 \, dx \\
&= \int \sin(x) \cos(x)^3 \, dx - \int \sin(x) \cos(x)^5 \, dx \\
&= \int u^5 \, du - \int u^3 \, du \\
&= -\frac{1}{4} u^4 + \frac{1}{6} u^6 + C = \frac{1}{6} \cos(x)^6 - \frac{1}{4} \cos(x)^4 + C
\end{aligned}
$$

$$
\begin{aligned}
u &= \cos(x) \\
du &= -\sin(x) \, dx
\end{aligned}
$$

$$\int \frac{1}{e^x + e^{-x}}\,dx = \int \frac{1}{e^x + e^{-x}}\,dx = \int \frac{e^x}{e^x}\,dx = \int \frac{e^x}{(e^x)^2 + 1}\,dx$$

So, let $u = e^x$. Then $du = e^x\,dx$ and so

$$\int \frac{1}{e^x + e^{-x}}\,dx = \int \frac{e^x}{(e^x)^2 + 1}\,dx = \int \frac{1}{u^2 + 1}\,du = \arctan(u) + C = \arctan(e^x) + C$$

$$\int (1+\sqrt[3]{x})^a \, dx = \int u^a \cdot 3 \cdot (u^2 - 2u + 1) \, du = \int 3u^{2+a} - 6u^{1+a} + 3u^a \, du$$
$$= \frac{3}{3+a} u^{3+a} - \frac{6}{2+a} u^{2+a} + \frac{3}{a+1} u^{a+1} + C$$
$$= \frac{3}{3+a}(1+\sqrt[3]{x})^{3+a} - \frac{6}{2+a}(1+\sqrt[3]{x})^{2+a} + \frac{3}{a+1}(1+\sqrt[3]{x})^{a+1} + C$$

Use $u = 1 + \sqrt[3]{x} = 1 + x^{\frac{1}{3}}$.

$$du = \frac{1}{3} x^{-\frac{2}{3}} \, dx$$
$$du \cdot 3 \cdot x^{\frac{2}{3}} = dx$$

Also notice that $(u-1)^3 = x$. Hence
$dx = du \cdot 3 \cdot x^{\frac{2}{3}} = du \cdot 3 \cdot ((u-1)^3)^{\frac{2}{3}} = du \cdot 3 \cdot (u-1)^2 = du \cdot 3 \cdot (u^2 - 2u + 1)$